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Mass flow through a circular orifice and a two-dimensional slit at high Knudsen numbers

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The local and total mass flow through a circular orifice and a two-dimensional slit with large upstream to downstream pressure ratio is calculated for large Knudsen number. The solution is obtained by performing one iteration on an integral form of the kinetic equation, starting with the free molecular solution for the distribution function. A relaxation model is used to describe the intermolecular collisions. There is found to be a variation of approximately 40 % in the local mass flow perturbation (from the free molecular value) across the orifice. For the slit the variations are smaller when the Reynolds number is less than 0·3, but are comparable for greater values. This Reynolds number, defined by $\mathrm{Re} = 2R\rho(kT/m)^{\frac{1}{2}}/\mu$ with R the radius of the orifice or half-width of the slit and with the flow field quantities evaluated far upstream, appears to be a less ambiguous quantitative parameter than the inverse Knudsen number. The ratio of total mass flow to the free molecular value is given by $[1 + 0{\cdot}083\,\mathrm{Re} + o(\mathrm{Re})]$ for the orifice and $[1 - 0{\cdot}057\,\mathrm{Re}\ln\mathrm{Re} + 0{\cdot}055\,\mathrm{Re} + o(\mathrm{Re})]$ for the slit. There is agreement within experimental error between the theoretical results and available experimental data for the circular orifice when $\mathrm{Re} < 1$.

1. Introduction

Liepmann (1961) has pointed out that the problem of flow through an orifice offers the possibility of a comparison between experiment and a kinetic theory analysis which should be insensitive to the nature of the interaction of the molecules with the physical boundaries present. Most of the other more widely analysed problems, e.g. plane Couette flow, give results which are crucially dependent on this interaction‡ for high Knudsen numbers. Liepmann has presented experimental results for the mass flow through a circular orifice. These experiments are limited to the case of large pressure ratios across the orifice. The discussion in this paper will also be limited to this case.

The mass flux for large Knudsen numbers can also be of interest when the molecules passing through the orifice are used as a molecular beam. In some experiments, e.g. when attempting to deduce accommodation coefficients for

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‡ The shock structure problem is of course a notable exception, but for this problem the accurate measurement of local properties of the gas is extremely difficult.

normal momentum by measuring the forces exerted upon objects placed in the beam (Stickney & Hurlbut 1963), a high molecular flux is desirable to increase the accuracy of the measurements. This flux increases with the pressure upstream of the orifice. However, the Knudsen number (based on a typical dimension of the orifice) decreases with increasing pressure and could fall to values where use of the classical free molecular value for the mass flow would no longer be justified. A correction to account for the effect of large but finite Knudsen number would then be of interest.

A theoretical calculation of the local mass flow at the centre of a circular orifice was given by Narasimha (1961). The calculation was based on a general iterative method of solving the kinetic equation for the distribution function (Willis 1958). This analysis was subject to certain restrictions and simplifying assumptions, outlined in § 2. It has been shown (Willis 1960) that the value of the centre-point mass flow is sensitive to the exact nature of any simplifying assumptions used.

In this paper we present the results of numerical calculations of the total mass flow through both a circular orifice and a two-dimensional slit. The calculations are restricted to one iteration and the effect of molecules which, due to collision, return through the orifice is neglected. No further simplifying assumptions, such as those used by Narasimha, are employed.

In § 2 we show that the use of only one iteration, starting from the free molecular distribution function, gives results for the mass flow of the form

$$\left.\begin{aligned} \dot{m}^{(1)} &= \dot{m}_{fm}[1 + a(\alpha) + o(\alpha)], \\ \alpha &\sim R/\lambda \sim 1/Kn, \end{aligned}\right\} \tag{1.1}$$

where R is the radius of the orifice or half-width of the slit, λ is the mean free path, and $a(\alpha) \sim \alpha$ for the orifice and $a(\alpha) \sim \alpha(\ln \alpha + c)$, where c is a constant of order unity, for the slit. Further iterations will in general change the value of the $o(\alpha)$ term, and, formally, there is no justification for retaining these terms. However, the calculation of these terms allows us to estimate the range of α over which the $o(\alpha)$ terms may be ignored and to estimate empirically the range of α for which the first iterate results should give reasonable agreement with experiment.

A comparison between the mass flows for the two geometries is also useful in estimating the possible effect of irregularities in small orifices which, due to fabrication difficulties, will rarely be exactly circular or two-dimensional.

2. The first iterate method

A complete kinetic theory analysis would call for a solution of the transport equation (the Maxwell–Boltzmann equation or a model equation) for arbitrary Knudsen number. Such a solution does not appear feasible at the present time and we restrict attention to obtaining a solution which is valid for high Knudsen number. We use the method of integral iteration proposed by Willis (1958). In this method the transport equation for steady state conditions with no external forces is written

$$\boldsymbol{\xi} . (\partial f/\partial \mathbf{s}) = -fD(f, \boldsymbol{\xi}) + P(f, \boldsymbol{\xi}), \tag{2.1}$$

where $f(\mathbf{s}, \boldsymbol{\xi})$ is the distribution function, $\mathbf{s}$ is the spatial co-ordinate (we do not

use the more conventional $\mathbf{r}$ in order that r may be used in the cylindrical co-ordinate scheme for the circular orifice), $\boldsymbol{\xi}$ the molecular velocity, P represents the production of molecules of velocity $\boldsymbol{\xi}$ due to collisions, and fD represents the loss of molecules of this class due to collisions. The equation can be written in this form for the Maxwell–Boltzmann equation, if the total collision cross-section is finite, and for the model equation used below. The iteration scheme is represented by

$$\boldsymbol{\xi}.(\partial f^{(n+1)}/\partial \mathbf{s}) = -f^{(n+1)}D(f^{(n)}, \boldsymbol{\xi}) + P(f^{(n)}, \boldsymbol{\xi}).$$

For high Knudsen number the obvious choice of $f^{(0)}$, the zeroth iterate, is the free molecular solution, i.e. the solution for infinite Knudsen number. The convergence of this iteration scheme has only been proved for three simple linearized problems (Willis 1958; Pao 1964). In general, however, an order of magnitude analysis indicates that if $M^{(n)}$ denotes a physically meaningful moment of $f^{(n)}$ then $|M^{(n+1)} - M^{(n)}|/|M^{(n)} - M^{(n-1)}|$ approaches zero as α approaches zero†. A similar statement regarding $f^{(n)}$ itself is true if 'small' regions in the vicinity of special values of $\mathbf{s}$ and $\boldsymbol{\xi}$ (frequently $\xi = 0$) are excluded. All moments are formally functions of α and can be expanded for small α in terms of elementary functions (usually powers of α and $\ln\alpha$). Certain leading terms in these expansions (often only the first term) will be the same for the first and subsequent iterations. Strictly, then, it is consistent to retain only these terms in the expansion of the first iterate moments. However, it is of interest to investigate the behaviour of the full first iterate as the differences between this and the leading terms in the expansion may be helpful in estimating the magnitude of changes to be expected from more iterations.

The formal solution of equation (2.1) for $f^{(1)}$ gives

$$f^{(1)}(\mathbf{s}, \boldsymbol{\xi}) = f(\mathbf{s}_0, \boldsymbol{\xi}) \exp\left(-\int_{\mathbf{s}_0}^{\mathbf{s}} \frac{D^0(\mathbf{s}'')}{\xi} ds''\right) + \int_{\mathbf{s}_0}^{\mathbf{s}} \frac{ds'}{\xi} P^0(\mathbf{s}') \exp\left(-\int_{\mathbf{s}'}^{\mathbf{s}} \frac{D^0(\mathbf{s}'')}{\xi} ds''\right), \tag{2.2}$$

where $\mathbf{s}$, $\mathbf{s}'$ and $\mathbf{s}''$ are all of the form $\mathbf{s}_0 + \boldsymbol{\xi}t$ and t is a positive scalar quantity. The notation $P^0(\mathbf{s}'')$ is merely shorthand for $P[f^{(0)}(\mathbf{s}'', \boldsymbol{\xi}), \boldsymbol{\xi}]$ and similarly for $D^0(\mathbf{s}'')$. The point $\mathbf{s}_0$ is arbitrary, but is obviously chosen to coincide with a point where the distribution function is known, i.e. at a boundary. The integrals in (2.2) are line integrals. A more convenient form for the present application is

$$f^{(1)}(\mathbf{s}, \boldsymbol{\xi}) = f(\mathbf{s}_0, \boldsymbol{\xi}) + \int_{\mathbf{s}_0}^{\mathbf{s}} \frac{ds'}{\xi} [P^0(\mathbf{s}') - f(\mathbf{s}_0, \boldsymbol{\xi}) D^0(\mathbf{s}')] \exp\left(-\int_{\mathbf{s}'}^{\mathbf{s}} \frac{D^0(\mathbf{s}'')}{\xi} ds''\right). \tag{2.3}$$

Narasimha (1961) has used essentially this form of the equation but derives it in a slightly different fashion.

Taking a co-ordinate system such that the orifice or slit lies in the plane $z = 0$,

† *Note added in proof.* However, recent results obtained in Pao (1964) indicate that the iteration scheme outlined above will have to be modified to take correct account of the non-uniform validity of the free molecular solution far from the orifice. Pao's result further indicates that the leading term given by integral iteration is correct.

and $z > 0$ corresponds to the upstream or high pressure side, the boundary conditions are

$$\left.\begin{aligned} & f(\mathbf{s}_0, \boldsymbol{\xi}) = n_\infty (h_\infty/\pi)^{\frac{3}{2}} \exp(-h_\infty \xi^2) \quad \text{for} \quad z \to +\infty, \quad \xi_z < 0, \\ \text{and} \quad & f(\mathbf{s}_0, \boldsymbol{\xi}) = 0 \qquad\qquad\qquad\qquad \text{for} \quad z \to -\infty, \quad \xi_z > 0, \end{aligned}\right\} \tag{2.4}$$

where n is number density and $h = m/2kT$. The point $\mathbf{s}$ is most conveniently taken in the plane $z = 0$, i.e. in the orifice or slit, and the local mass flow is given by

$$\dot{m}^{(1)} = \iiint -(m\xi_z) f^{(1)}_{(z=0)} d^3\xi, \tag{2.5}$$

where strictly the integration should be over all values of $\boldsymbol{\xi}$, but neglecting the molecules scattered back through the orifice or slit we may restrict the integral to negative values of ξ_z.

It can be shown that D^0 is of order $\bar{\xi}/\lambda$ where λ is the mean free path, and $[P^0(\mathbf{s}'') - D^0(\mathbf{s}'') f_\infty]$ is of order $(\bar{\xi} f_\infty/\lambda)\,\Omega(\mathbf{s}'')$, where $\Omega(\mathbf{s}'')$ is the solid angle subtended at $\mathbf{s}''$ by the orifice or the slit. The mass flow then becomes

$$\left.\begin{aligned} \dot{m}^{(1)} &= \dot{m}_{fm}[1 + a_1\alpha + o(\alpha)] \qquad \text{(orifice)}, \\ \dot{m}^{(1)} &= \dot{m}_{fm}[1 + a_2\alpha \ln\alpha + a_3\alpha + o(\alpha)] \quad \text{(slit)}, \end{aligned}\right\} \tag{2.6}$$

where $\alpha \sim R/\lambda$ is an inverse Knudsen number and a_1, a_2 and a_3 are constants of order unity. The correctness of these expressions is most readily seen from the specific analysis below using Krook's model (Bhatnager, Gross & Krook 1954) for the collision process, but we believe them to be correct for any kinetic equation where splitting the collision operator into the form $P - fD$ is valid.

The solution, as represented by equations (2.3) to (2.5), is now given by a series of quadratures. However, the calculation of P^0 and D^0, even for such a simple molecular model as rigid spheres, would be an onerous task, and we effect a great simplification by replacing the Maxwell–Boltzmann collision operator by the simple statistical operator suggested by Krook (Bhatnager *et al.* 1954). In this model D and P are given by the simple expressions

$$D = \delta n, \quad P = \delta n^2 (h/\pi)^{\frac{3}{2}} \exp[-h(\boldsymbol{\xi} - \mathbf{u})^2], \quad h = m/2kT, \tag{2.7}$$

where δ is a constant, and n, $\mathbf{u}$ and T are the local number density, macroscopic velocity and temperature, defined by

$$(n, n\mathbf{u}, nkT) = \iiint f\left(1, \boldsymbol{\xi}, \frac{m(\boldsymbol{\xi} - \mathbf{u})^2}{3}\right) d^3\xi. \tag{2.8}$$

We define the following non-dimensional quantities

$$\left.\begin{aligned} N &= n^0/n_\infty, \quad B = h^0/h_\infty = T_\infty/T^0, \quad C = h_\infty^{\frac{1}{2}}\,\xi; \\ \mathbf{W} &= h_\infty^{\frac{1}{2}}\,\mathbf{u}, \quad \eta' = |\mathbf{s} - \mathbf{s}'|/R, \qquad \alpha = \delta n_\infty h_\infty^{\frac{1}{2}} R, \end{aligned}\right\} \tag{2.9}$$

where the superscript 0 denotes the free molecular quantities. All of these with the exception of α will in general be of order unity. The parameter α has dimensions of inverse Knudsen number and is usually a small quantity. The actual relationship between α and R/λ_∞ can be assigned in several ways, as is discussed in §4.

Substituting equations (2.3), (2.4), (2.7) and (2.8) in (2.5) and using the fact that $\dot{m}_{fm} = mn_\infty/(2\pi^{\frac{1}{2}}h_\infty^{\frac{1}{2}})$, we obtain

$$\dot{m}^{(1)}(\mathbf{s}) = \dot{m}_{fm}\left[1 - \frac{2\alpha}{\pi}\iiint_{C_z<0} d^3C\frac{C_z}{C}\int_0^\infty d\eta' N(\eta')\,Z(\eta')\right.$$
$$\left.\exp\left(-\alpha\int_0^{\eta'}\frac{N(\eta'')}{C}d\eta''\right)\right], \quad (2.10)$$

where $Z(\eta') = N(\eta')\,[B(\eta')]^{\frac{3}{2}}\exp\{-B(\eta')\,[\mathbf{C}-\mathbf{W}(\eta')]^2\} - \exp(-C^2)$.

Because the vectors $\mathbf{s}'-\mathbf{s}_0$, $\mathbf{s}''-\mathbf{s}_0$ and $\mathbf{s}-\mathbf{s}_0$ are parallel, the functions $N(\eta')$, etc., depend in fact upon the co-ordinates of $\mathbf{s}$, i.e. the position in the orifice, and the direction of $\mathbf{C}$ as well as η'. By observing that $|N(\eta')-1|$, $|B(\eta')-1|$ and $|W(\eta')|$ are all of order $(\eta')^{-2}$ for the orifice and $(\eta')^{-1}$ for the slit when $\eta' \gg 1$ the results of equation (2.6) are readily verified.

For the slit one of the quadratures (that with respect to the component of $\mathbf{C}$ parallel to the edges of the slit), can be performed analytically, and for the circular orifice one integral is trivial when $\mathbf{s}$ is at the centre of the orifice. Further quadratures require numerical work or approximation of the integrand. Narasimha's calculation was limited to the centre-point mass flow with the exponential factor explicitly demonstrated in the integrand of equation (2.10) set equal to unity. This preserves the leading term, which is of order α, but neglects all higher-order terms. Narasimha further simplified the calculation by assuming analytic expressions for N, B and $\mathbf{W}$ which have the correct behaviour along the centre line of the orifice and at large distances from the orifice.

Narasimha's result, in the notation of this paper, is (subscript c denotes centre line)

$$\dot{m}_c^{(1)} = \dot{m}_{fm}(1 + a\alpha),$$

with $a = 0{\cdot}22$. Willis (1960) obtained a value of $a = 0{\cdot}28$ by determining N, B and $\mathbf{W}$ numerically, and values of a ranging from $0{\cdot}02$ to $0{\cdot}24$ by using different analytic approximations to N, B and $\mathbf{W}$, which, at least *a priori*, appeared as plausible as those used by Narasimha. In view of these results it was decided to proceed with a numerical evaluation of the quadratures in equation (2.9) without making any further approximations. The only quadrature that offers more than routine difficulties is, for both the orifice and the slit, of the form

$$I_n(a, b) = \int_0^\infty x^n \exp(bx - a/x - x^2)\,dx.$$

The parameter b has the order of the local Mach number, a can take on all positive values, and x represents a non-dimensional velocity (the actual velocity for the orifice, and the component in the plane perpendicular to the edges for the slit). This integral is discussed in a more detailed report (Willis 1964).

3. Free molecular flow field

To perform the quadratures in equation (2.9) we must calculate the free molecular flow field quantities n^0, T^0, and $\mathbf{u}^0$. We use the definitions in equation (2.8), together with the free molecular solution. Upstream of the orifice (or slit)

this is given by $f_{fm}(\mathbf{s}, \boldsymbol{\xi}) = 0$ if the vector $\boldsymbol{\xi}$ lies within the solid angle subtended by the orifice at $\mathbf{s}$ and is directed away from the orifice and $f_{fm}(\mathbf{s}, \boldsymbol{\xi}) = f_\infty$ for all other values of $\boldsymbol{\xi}$.

For both the orifice and slit it readily follows that (Narasimha 1961)

$$n^0/n_\infty = N(\mathbf{s}'') = [1 - \Omega(\mathbf{s}'')/4\pi], \tag{3.1}$$

$$T^0/T_\infty = 1/B(\mathbf{s}'') = 1 - m(u^0)^2/3kT_\infty = (1 - 2W^2/3), \tag{3.2}$$

where Ω is the solid angle subtended by the orifice or slit. The velocities $\mathbf{u}^0$ and the solid angles take different forms for the two geometries.

For the orifice we use cylindrical co-ordinates (r, θ, z) with the origin at the centre of the orifice (now given by $z = 0$, $r \leqslant R$). The solid angle is given by

$$\Omega(r, z) = 2\pi + z \int_0^{2\pi} \frac{d\phi (Rr\cos\phi - z^2 - r^2)}{(R^2 + z^2 + r^2 - 2Rr\cos\phi)^{\frac{1}{2}} (z^2 + r^2\sin^2\phi)}. \tag{3.3}$$

This expression can be arranged in terms of elliptic integrals. We evaluated it using a scheme proposed by Naito (1957), who also gives references to previous investigations and tables available in the literature. A detailed discussion is given in Willis (1964). The non-dimensional velocity components are given by the following formulas:

$$NW_z = -0{\cdot}25\pi^{-\frac{1}{2}}[1 + (R^2 - r^2 - z^2)/D], \tag{3.4}$$

$$NW_r = -\pi^{-\frac{1}{2}} R^2 zr/[D(D + R^2 + r^2 + z^2)], \tag{3.5}$$

where

$$D^2 = (R^2 + r^2 + z^2)^2 - 4r^2R^2.$$

These formulas were derived by straightforward, if rather tedious, integration. A partial check on the algebra is provided by Narasimha's results for $r = 0$ and for $(r^2 + z^2) \gg R^2$.

For the two-dimensional slit we use Cartesian co-ordinates (x, y, z) such that the slit is given by $z = 0$, $-R \leqslant x \leqslant R$. The solid angle is given by

$$\Omega(x, z) = 2(\theta_2 - \theta_1), \tag{3.6}$$

where

$$\tan\theta_1 = z/(x + R), \quad \tan\theta_2 = z/(x - R), \tag{3.7}$$

and we take both θ_1 and θ_2 in the range 0 to π.

The velocity components are now given by

$$NW_x = -0{\cdot}25\pi^{-\frac{1}{2}} (\sin\theta_2 - \sin\theta_1), \tag{3.8}$$

$$NW_z = -0{\cdot}25\pi^{-\frac{1}{2}} (\cos\theta_1 - \cos\theta_2). \tag{3.9}$$

These relations are all exact and can be obtained from the definitions and elementary integrals.

All the information needed to perform the quadratures in equations (2.10) is now available.

4. Results

The local mass flow, using the complete first iterate, was calculated for five points across the orifice or slit and for six values of the rarefaction parameter. To give comparable weighting for each point in integrating over the orifice or slit to obtain the total mass flow we chose the points x/R (slit) or r^2/R^2 (orifice) to be equal to 0·00, 0·25, 0·50, 0·75, and 1·00. For the rarefaction parameter we used

the values $(\pi^{\frac{1}{2}}\alpha/2) = \epsilon = 0{\cdot}0,\ 0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}5,\ 1{\cdot}0$, and $2{\cdot}0$ (ϵ is the value of R/λ suggested originally by Narasimha). We estimate that there is less than 1% error in the numerical evaluation of $(\dot{m}^{(1)} - \dot{m}_{fm})$.

For the slit in the limit of very small α the quadratures can be performed analytically. The mass flow correction is independent of x/R and given by the formula

$$\dot{m}^{(1)} = \dot{m}_{fm}[1 - (\tfrac{1}{4}\pi^{\frac{1}{2}} - \tfrac{1}{2}\pi^{-\frac{1}{2}})\,\alpha \ln \alpha + O(\alpha)]$$
$$= \dot{m}_{fm}[1 - 0{\cdot}161\alpha \ln \alpha + O(\alpha)]. \tag{4.1}$$

The $O(\alpha)$ term is, however, dependent on x/R.

Before proceeding with the discussion we must relate α to some physically meaningful quantities. Narasimha has expressed α in terms of R/λ by equating the number of collisions (per unit volume and time) in the undisturbed gas according to Krook's model, namely, $\frac{1}{2}\delta n_\infty^2$, to $\frac{1}{2}n_\infty \bar{\xi}/\lambda$. Using $\bar{\xi} = 2/(\pi h_\infty)^{\frac{1}{2}}$, this gives

$$\alpha = \delta n_\infty h_\infty^{\frac{1}{2}} R = (2/\pi^{\frac{1}{2}})\,(R/\lambda). \tag{4.2}$$

However, the experimental values of mass flow are presented in terms of a Reynolds number defined by the experimentally determined viscosity, density and temperature. Specifically, Narasimha uses

$$\mathrm{Re} = 2R\rho_\infty/\mu_\infty(2h_\infty)^{\frac{1}{2}}. \tag{4.3}$$

For the Krook model the Chapman–Enskog viscosity is given by $\mu_\infty = p_\infty/\delta n_\infty$ and hence, after a little algebra,

$$\mathrm{Re} = 2\sqrt{2}\,\alpha. \tag{4.4}$$

If we choose to define a mean free path by the common formula

$$\lambda = (2\mu)/(\rho\bar{\xi}),$$

we find

$$\alpha = (\tfrac{1}{2}\pi^{\frac{1}{2}})\,(R/\lambda). \tag{4.5}$$

We believe that equation (4.4) is most appropriate for comparison between theory and experiment. The use of Knudsen number can be ambiguous, as shown by equations (4.2) and (4.5). We, therefore, present all results in terms of Re.

The local mass flow for the orifice is shown as a function of Reynolds number and position in figure 1. The curve corresponding to $\mathrm{Re} = 0$ gives the leading term for this case. The differences between this curve and those for non-zero values of Re show the difference between the full first iterate and the leading term results.

It is immediately apparent that there is a significant variation in the mass flow perturbation across the orifice. The average mass flow is always at least 20% lower than the local centre-point mass flow. Narasimha's approximate calculation of the centre-point mass flow gives, with our interpretation of Re,

$$[(\dot{m}_c^{(1)}/\dot{m}_{fm}) - 1] = 0{\cdot}078\,\mathrm{Re}, \tag{4.6}$$

compared to the correct value, for $\mathrm{Re} \to 0$, of $0{\cdot}104\,\mathrm{Re}$. The leading term for the total mass flow is given by

$$[(\dot{m}^{(1)}/\dot{m}_{fm}) - 1]_{\mathrm{total}} = 0{\cdot}083\,\mathrm{Re}, \tag{4.7}$$

so that Narasimha's estimate of the total mass flow, which was obtained by assuming that the mass flow perturbation is uniform across the orifice, and hence is given by the same expression as in equation (4.6), is in fact quite good. The

errors introduced by the approximations Narasimha used are to a large extent compensatory.

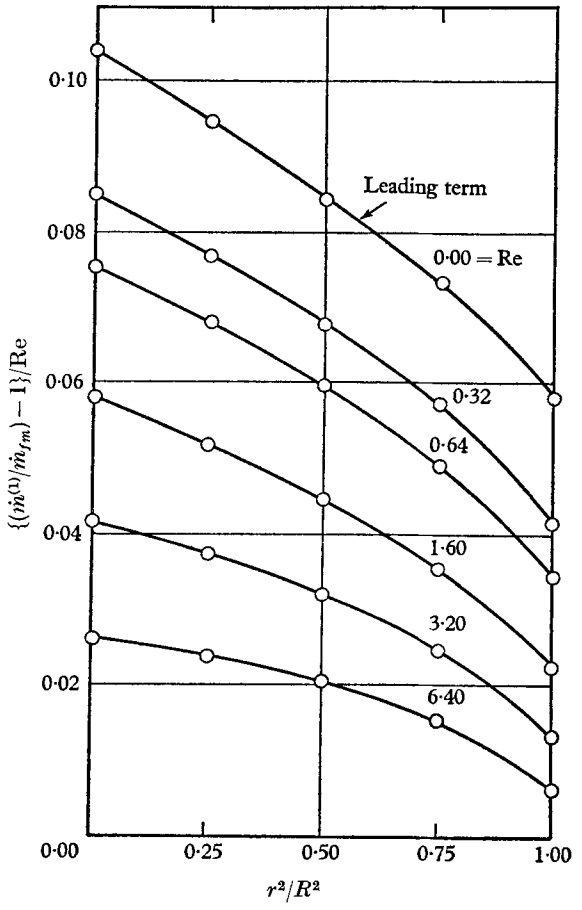

FIGURE 1. Local mass flow perturbation for circular orifice. Full first iterate results.

$$\mathrm{Re} = 2R\rho(kT/m)^{\frac{1}{2}}/\mu.$$

The local mass flow for the slit is shown as a function of Reynolds number and position in figure 2. The leading term for this case is of the form

$$(\dot{m}^{(1)}/\dot{m}_{fm}) = 1 + \mathrm{Re}[-0{\cdot}057 \ln \mathrm{Re} + c(x/R)], \tag{4.8}$$

where $c(x/R)$ is a function of order unity. This cannot be shown on the figure with our choice of the non-dimensional ordinate. Again it can be seen that, except for very small Re, there is an appreciable difference in the mass flow perturbation from the centre to the outside of the slit.

The total mass flow is shown as a function of Reynolds number for both geometries in figure 3. For the orifice the leading term is given by equation (4.7), and the error in the numerical value of 0·083 is estimated to be less than 1 %. For the slit the leading term for the total mass flow perturbation is

$$[(\dot{m}^{(1)}/\dot{m}_{fm}) - 1]_{\mathrm{total}} = -0{\cdot}057\,\mathrm{Re} \ln \mathrm{Re} + \bar{c}\,\mathrm{Re}. \tag{4.9}$$

The constant $\bar{c}$ cannot be calculated to the same accuracy as the constant $-0{\cdot}057$ without performing a special computation. However, it was found that the expression

$$[(\dot{m}^{(1)}/\dot{m}_{fm}) - 1]_{\text{total}} = -0{\cdot}057\,\text{Re}\ln\text{Re} + 0{\cdot}055\,\text{Re} + 0{\cdot}022\,\text{Re}^2 - 0{\cdot}005\,\text{Re}^2\ln\text{Re} \qquad (4.10)$$

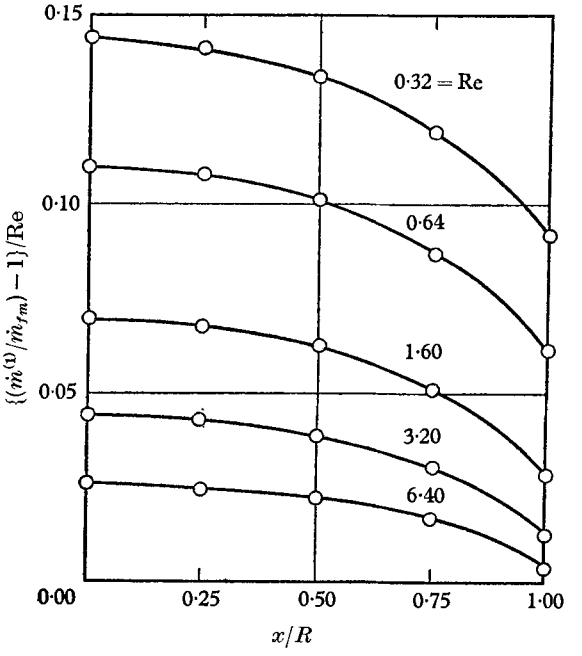

FIGURE 2. Local mass flow perturbation for two-dimensional slit. Full first iterate results. $\text{Re} = 2R\rho(kT/m)^{\frac{1}{2}}/\mu$.

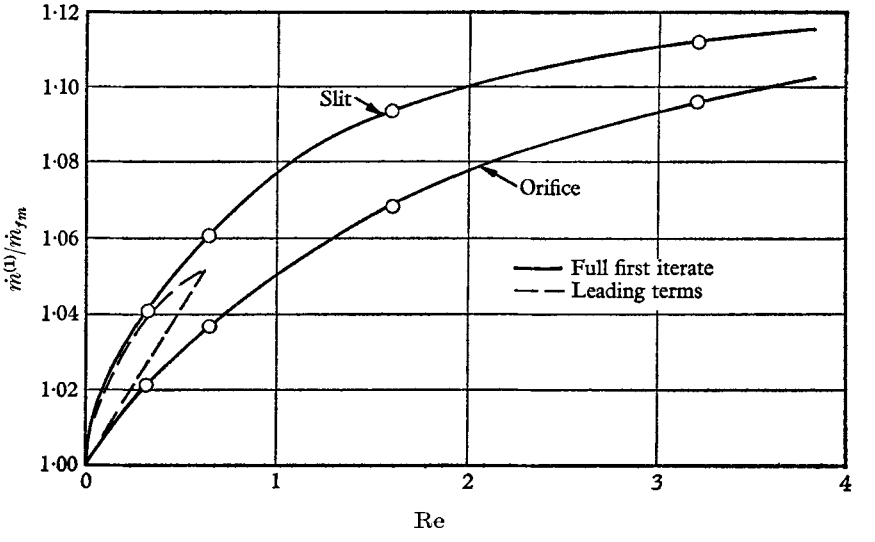

FIGURE 3. Comparison of total mass flow for circular orifice and two-dimensional slit. $\text{Re} = 2R\rho(kT/m)^{\frac{1}{2}}/\mu$.

fits the numerical values of $(\dot{m}^{(1)}/\dot{m}_{fm})_{\text{total}}$ for Re = 0·32, 0·64 and 1·60, and therefore a reasonable estimate for $\bar{c}$ is 0·055. This value was used in plotting the leading term in figure 3.

It can be seen that there are significant differences predicted for the two geometrics. However, the choice of orifice radius and half-width of the slit as the characteristic lengths is somewhat arbitrary. We might, for example, select as an effective dimension the ratio of area to length of perimeter, in which case the effective Reynolds number for the orifice would be halved and the differences between the two geometries would appear much less.

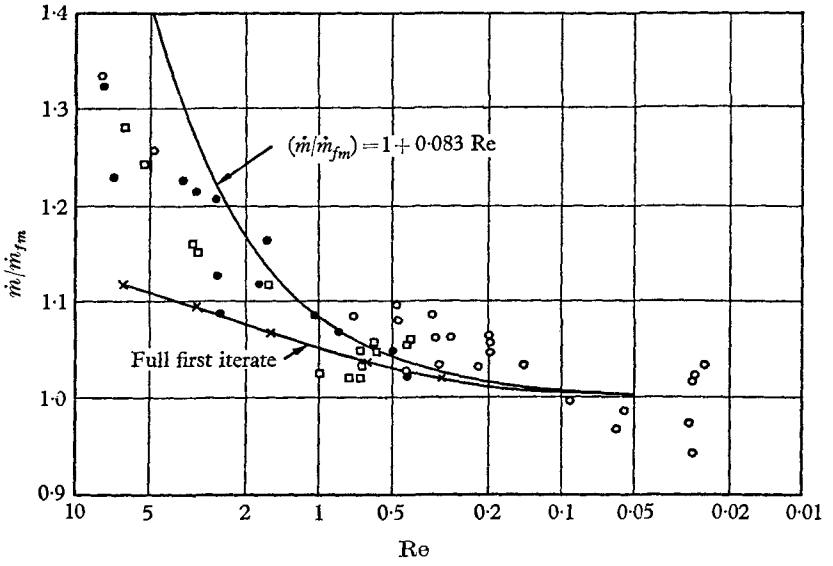

FIGURE 4. Comparison of theory and Liepmann's data. ×, Computed values from full first iterate. Data: ○, helium; ●, argon; □, nitrogen. $\mathrm{Re} = 2R\rho(kT/m)^{\frac{1}{2}}/\mu$.

A comparison between theory and experiment for the circular orifice is shown in figure 4. The only strictly valid comparison is one between the leading term result [equation (4.7)] and the experimental data for small values of α the 'natural' small parameter for the problem. Bearing in mind the numerical factor in equation (4.4), a liberal upper limit for Re would be unity. Unfortunately, for $\mathrm{Re} \leqslant 1$ the relative spread in the experimental data is too large to draw any definite conclusions. At best we may claim fair agreement between either the leading term or the complete first iterate results and the experimental data. (The agreement looks better if we disregard the very scattered helium data.) For $\mathrm{Re} > 1$ there is no reason to expect that either theoretical result should agree with the data. Indeed, we see that the full first iterate results clearly underestimate the mass flow, and the linear expression overestimates the mass flow. It is not unreasonable to hope that further iterations (which certainly should be called for, as there is no justification for expecting the first iterate correction to be dominant unless the Reynolds number is small) would improve the agreement between theory and experiment.

Probably coincidentally, the full first iterate results for the *local* mass flow at the centre of the orifice agree rather well with the experimental results for the total mass flow. This was noted by Sherman (1963).

5. Conclusions

The problems considered are sufficiently simple that quite detailed theoretical calculations can be made without introducing (possibly crucial) simplifying assumptions about the free molecular distribution function.

For the circular orifice there is fair agreement between theory and experiment. A more definitive comparison can only be made if the experimental scatter can be greatly reduced for low values of the Reynolds number (i.e. large values of the Knudsen number), or if the theory can be extended to higher values of the Reynolds number, perhaps by performing another iteration in the proposed scheme. For the circular orifice the local mass flow perturbation is greatest at the centre and falls to approximately 60 % of the centre value at the edge. This is true for all Reynolds numbers investigated. For the slit, the variation decreases with decreasing Reynolds number, but is comparable to that of the orifice for Reynolds number greater than about 0·3. Use of the centre-point value to estimate the total mass flow (Narasimha 1961) would therefore give too large values of the mass flow. However, Narasimha's approximate calculation of the centre-point mass flow perturbation, to order Re, is about 20 % low, and his resulting estimate of the total mass flow is quite accurate.

There are significant differences between the theoretical predictions for the mass flow perturbation for a circular orifice and a two-dimensional slit, but these differences might be minimized by changing the rather arbitrary characteristic lengths used in defining the Reynolds number.

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